



# Reversible Thinking of Fifth Graders: Focus on Linier Equations

Syarifatul Maf'ulah<sup>1\*</sup>

<sup>1</sup> Pendidikan Matematika, STKIP PGRI Jombang, Indonesia

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## ABSTRAK

Berpikir reversibel adalah kemampuan mental untuk mengembalikan cara berpikir ke titik awal. Penelitian ini bertujuan untuk mengeksplorasi pemikiran reversibel siswa kelas lima pada persamaan linier. Penelitian deskriptif-eksploratif ini dilakukan terhadap 96 siswa kelas lima sekolah dasar. Pengumpulan data dilakukan dengan memberikan tugas berpikir reversibel dan wawancara. Teknik yang digunakan untuk analisis data yaitu analisis deskripsi kualitatif dan kuantitatif. Tugas berisi sebuah persamaan linier (disebut persamaan awal). Siswa diminta untuk membuat persamaan lain yang senilai dengan persamaan awal. Semua persamaan yang berhasil dibuat siswa kemudian dianalisis dan diklasifikasikan berdasarkan strategi yang digunakan. Selanjutnya dipilih satu siswa dengan strategi paling beragam untuk dieksplorasi lebih dalam pemikirannya. Hasil penelitian, terdapat 10 kategori strategi berpikir reversibel siswa, antara lain pertama, memindah elemen pembangun persamaan awal. Kedua, menentukan elemen yang tidak diketahui. Ketiga, mengoperasikan kedua ruas persamaan awal dengan bilangan yang sama. Keempat, mengoperasikan kedua ruas persamaan awal dengan ekspresi yang bernilai sama. Kelima, mengubah elemen pembangun yang diketahui pada persamaan yang dibuat. Keenam mengacu pada nilai dari elemen yang tidak diketahui kemudian mengubah nilai tersebut menjadi suatu ekspresi. Ketujuh, mengubah elemen pembangun yang diketahui pada persamaan awal. Kedelapan, mengacu pada nilai variabel persamaan awal. Kesembilan, menggunakan definisi pengurangan. Kesepuluh, menggunakan sifat komutatif terhadap operasi jumlah.

## ABSTRACT

Reversible thinking is the mental ability to return the way of thinking to the original point. This study explores the reversible thinking of fifth graders on linear equations. This descriptive-exploratory research was conducted on 96 fifth-grade elementary school students. Data collection is done by providing reversible thinking and interviews. The technique used for data analysis is qualitative and quantitative description analysis. The assignment contains a linear equation (called the initial equation). Students are asked to make an equation equivalent to the initial equation. All equations that students successfully created were then analyzed and classified based on the strategies used. Next, one student with the most diverse approach was selected to be explored more deeply in reversible thinking. The results of the study, there are ten categories of students' reversible thinking strategies, among others, first, moving the elements of the initial equation builder. Second, determine the unknown factors. Third, operate both sides of the initial equation with the same number. Fourth, operate both as equations of equivalent expressions. Fifth, change the known building elements in the created equation. Sixth refers to the value of the unknown component then converts that value into a presentation. Seventh, transforming the building elements that are known in the initial equation. Eighth refers to the value of the initial equation variable. Ninth, using the definition of subtraction. Tenth, using the commutative property of the operation of the number.

## 1. INTRODUCTION

Education is a conscious and planned knowledge transfer process to change human behavior and mature humans through the teaching process of informal, non-formal, and informal education (Eze et al., 2018; Lindvig & Mathiasen, 2020; Sharma & Srivastava, 2020). Education is an essential thing and cannot

be separated in the life of the nation and the state. Through education, students can think in a reversible way. Thinking reversible or often known as 'reversibility' is a kind of capability to think as introduced by Piaget. This research was inspired by Piaget's theory on reversibility. Piaget argued that one characteristic of a concrete stage was the development of reversibility (Inhelder & Piaget, 1958; Muzaini et al., 2021). When reversibility was considered as a feature of the development phase of one's cognition, it was definitely crucial. Relating to mathematics education, reversibility is a part of mathematical competence influencing students' successfulness in solving mathematics problems (Daulay et al., 2019; S. Mafulah & Juniati, 2019; Zandieh & Rasmussen, 2010). Besides, problem-solving is the heart of learning mathematics (Hendriana et al., 2018; Setiyani et al., 2020; Sumirattana et al., 2017). Therefore, mathematical reversible thinking contributes to students' competence on mathematics problem-solving and it should be put into account (Silwana et al., 2020; Surya et al., 2017). In addition, the competence of reversible thinking is also considered as one characteristic of one's intelligence.

Reversible thinking was considered as the main prerequisites for many problems at each level of mathematics (Fitmawati et al., 2019; Hackenberg, 2010). Because reversible thinking will encourage one's reasoning. In reversible thinking, there are two opposite mental activities (Hafner et al., 2012; Sutiarsa, 2020). Two activities that can strengthen one's reasoning, while reasoning is needed in every mathematical problem solving. Reversibility is one's competence to build a two-way relationship; that is from an initial point to the expected goal and from that reached goal back to the initial point. When reversibility occurs, two processes may exist: a process from an initial point (before a mental operation happened) to the expected goal (after a mental operation happened), and a process from that reached goal back to the initial point. This research primarily focused on primary graders' mathematical reversible thinking with a consideration that students' reversible thinking began to develop since they were 7 years old up to 11 years old.

One subject matter that may help to explore primary graders' mathematical reversibility is arithmetic-algebra, by providing a test containing arithmetic-algebra problems (Arnau et al., 2013; Humberstone & Reeve, 2018). Arithmetic and algebra were material tightly related to mathematical reversible thinking (Ábrahám et al., 2021; Buium & Miller, 2020). Algebra taught to primary graders is known as Early-Algebra or arithmetic-algebra. Early-algebra or arithmetic-algebra is not only used as means to learn algebra in further level, but also helpful to develop students' conceptual thinking on mathematics to be more comprehensive and complex since early ages (Humberstone & Reeve, 2018; McMullen et al., 2017). One form of Early-algebra that can be given to fifth graders is a one-variable linear equation, so that the focus of students' reversible thinking in this study on linear equations. An equation is a mathematical statement in which a sign for equality "=" is used to show an equivalence between the number or expression on the left side and the number or expression on the right side. Expression is itself defined as a combination between an (operand) number and an arithmetical operation without involving the sign "=" within (Powell, 2012). Furthermore, Powell argued that a mathematical equation is an equation with zero or one variables, whereas an algebraic equation is an equation with two or more variables. In relation to the equation given to primary graders, this study pointed to mathematical equation since it has one variable i.e. linear equation.

Reversible thinking is defined as a primary grader's mental activity to make particular equations equivalent with the original one and then reverse them to the initial (Mafulah et al., 2019; S. Mafulah & Juniati, 2019, 2020). In this case, the given linear equation is set as an initial point. The primary graders are asked to make as many as equations equivalent to the given one. Two equations are considered equivalent if they have similar solution. The equations made by the students are set as the reached goal. Thus, mathematical reversible thinking contains two fundamental characteristics. (1) A mental process from an initial point to the expected goal. An initial point is the original equation provided on the task sheet. The primary graders are asked to make as many possible as new equations which are equivalent with the original one. The new equations the students may make are the expected goal. The process of making new equations equivalent with the original one, in this study, is considered as students' steps and strategies to make equations. (2) A process of reverting from the reached goal to the initial point. If the initial point considered in this study is the original equations given on the task sheet and the expected goal is the new equation (which is equivalent with the original one), the second characteristic of mathematical reversible thinking points the students' process in reversing the new equation to the initial point, and thus, it refers to the primary students' steps and strategies to return the new equation into the initial point. This study aims to explore the reversible thinking of fifth graders on linear equations.

## 2. METHOD

This descriptive-exploratory study was conducted on 96 fifth grade elementary school students in Jombang, East Java. Data was collected by giving reversible thinking tasks and interviews. The task

contained a linier equation and the students were asked to make as many as equations equivalent to the given one. All equations the students successfully made were then analyzed and classified based on the strategy used. An linier equation was given at the task was  $24 + a = 16$ . Furthermore, the result of analysis would be then described. Among all the students, one student with the most variety of strategies was selected to be interviewed and his reversible thinking was then explored based on the process and the strategy he used to make new equations and revert them to the original one. The result of the task in Table 1.

**Table 1.** The Result of Mathematical Reversible Thinking of Primary Graders

Strategy	Strategy to Make Equations	The Number of Students		Total	Percentage
		Male	Female		
1	Moving the constructing element of an initial equation	33	45	78	81.3 %
2	Determining unknown elements	25	36	61	63.5 %
3	Operating the both sides of the initial equation by using the same number	8	2	10	9.6 %
4	Operating the both sides of the initial equation by using equal expressions	5	-	5	4.8 %
5	Making equations based on the previously made one, and then changing the identified constructing element.	7	5	12	12.5 %
6	Pointing to the value of the unknown element and then turning it into an expression.	20	33	53	55.2 %
7	Pointing to the initial equation and then changing the identified constructing element.	15	17	32	33.3 %
8	Making any kinds of equations in case that the value of the unknown element equals to the value of the unknown element on the initial equation.	10	7	17	17.7 %
9	Using the definition of subtraction.	1	-	1	1.01%
10	Using the feature of commutative rule on the operation of addition.	5	1	6	6.25 %
11	Trial and error	31	2	33	34.4 %
12	Pointing to the previously-made equation and then changing the position of the both sides of equation.	2	-	2	2.1 %
13	Moving the constructing element without using any clear procedures.	5	18	22	22.9 %

Based on Table 1, it informs that there are 13 categories of strategies the students used to make equations. The subject selected here is the student who used the most variety of strategies to make equations. He's code name is D. He found using category of strategy 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. The next step was having a close interview with D in order to explore his mathematical reversible thinking, which included: (1) the process of making equations which are equivalent to the initial equation, it involves the procedures and strategies used; (2) the process of reversing the D's equations to the initial equation, it involves the procedures and strategies used to reverse the equations to the initial one. The technique used to analyze the data is descriptive qualitative and quantitative analysis.

### 3. RESULT AND DISCUSSION

#### Result

The result of task was done by D was illustrated by Figure 1.

**Lembar Penyelesaian**

<p>① <math>a = 16 - 24</math></p> <p>② <math>a = -8</math></p> <p>③ <math>-24 + 16 = a</math></p> <p>④ <math>a + 24 = 16</math></p> <p>⑤ <math>a = 24 - (16 \times 2)</math></p> <p>⑥ <math>-24 : 2 + 16 - 12 = a</math></p> <p>⑦ <math>a = (-1) \times 8</math></p> <p>⑧ <math>a = -24 : 3</math></p> <p>⑨ <math>(20 + 4 - (16 \times 4)) : 5 = a</math></p> <p>⑩ <math>3a + 40 = 16</math></p> <p>⑪ <math>22 + 2 + a = 15 + 1</math></p> <p>⑫ <math>24 + a = 10 + 6</math></p> <p>⑬ <math>10 + 6 + a + 16 = 24</math></p> <p>⑭ <math>24 + a - 6 + 10 - 4 = 16</math></p> <p>⑮ <math>24 + 16 + 20 - 8 = 9 + 7</math></p> <p>⑯ <math>0 + a = -8</math></p> <p>⑰ <math>8 + 2a = -8</math></p> <p>⑱ <math>20 + 4 + a = 12 + 4</math></p> <p>⑲ <math>19 + 5 + a = (16 \times 2) : 2 = 0</math></p> <p>⑳ <math>18 + 6 + a = 14 + 2</math></p> <p>㉑ <math>17 + 7 + a = 11 + 3 + 2</math></p>	<p>⑳ <math>15 + 9 + a = 10 + 7 - 1</math></p> <p>㉓ <math>23 + 1 + a = 15 + 5 - 4</math></p> <p>㉔ <math>8 \times 3 + a = 8 \times 2</math></p> <p>㉕ <math>10 + 14 + a = 10 + 10 + 10 - 14</math></p> <p>㉖ <math>5 \times 16 - (24 + a) \times 5</math></p> <p>㉗ <math>5 + 16 = 24 + a + 5</math></p> <p>㉘ <math>(1 : 1 \times 5 - 2) + (24 + a) = 16 + (5 - 2 \times 1 : 1)</math></p> <p>㉙ <math>a = 24 + 16 - (50 - 2)</math></p> <p>㉚ <math>20 - (24 + a) = 20 - 16</math></p> <p>㉛ <math>-10 + 2 = a</math></p> <p>㉜ <math>-20 + 25 = 5 + a + 8</math></p> <p>㉝ <math>-8 + 8 = a + 8</math></p> <p>㉞ <math>10 + 6 - (20 + 4) = a</math></p> <p>㉟ <math>(8 \times 2) - 24 = a</math></p> <p>㊱ <math>25 - 1 + a = 32 : 2</math></p>
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Figure 1. The Result of the Subject's Mathematical Reversible Thinking

In order to facilitate the analysis, the researcher codified the subject's equations as seen in the following Table 2.

Table 2. Codifying the Result of Reversible-Thinking Task

Code	D's Equations
P1	$a = 16 - 24$
P2	$a = -8$
P3	$-24 + 16 = a$
P4	$a + 24 = 16$
P5	$a = 24 - (16 \times 2)$
P6	$-24 : 2 + 16 - 12 = a$
P7	$a = (-1) \times 8$
P8	$a = -24 : 3$
P9	$(20 + 4 - (16 \times 4)) : 5 = a$
P10	$3a + 40 = 16$
P11	$22 + 2 + a = 15 + 1$
P12	$24 + a = 10 + 6$

Code	D's Equations
P13	$10+6+a+16=24$
P14	$24+a-6+10-4=16$
P15	$24+16+20-8=9+7$
P16	$0+a=-8$
P17	$8+2a=-8$
P18	$20+4+a=12+4$
P19	$19+5+a=(16 \times 2):2=0$
P20	$18+6+a=14+2$
P21	$17+7+a=11+3+2$
P22	$15+9+a=10+7-1$
P23	$23+1+a=15+5-4$
P24	$8 \times 3+a=8 \times 2$
P25	$10+14+a=10+10+10-14$
P26	$5 \times 16-(24+a) \times 5$
P27	$5+16=24+a+5$
P28	$(1:1 \times 5-2)+(24+a)=16+(5-2 \times 1:1)$
P29	$10=24+16-(50-2)$
P30	$20-(24+a)=20-16$
P31	$-10+2=0$
P32	$-20+25=5+a+8$
P33	$-8+8=0+8$
P34	$10+6-(20+4)=0$
P35	$(8 \times 2)-24=0$
P36	$25-1+a=32:2$

Note: Pi = i<sup>th</sup> equation, with i = 1, 2, ..., 36

Based on the result of interview, it shows that the subject used 10 strategies of reversible thinking on linier equation as follow.

**Moving the constructing element**

The equation classified into this category is coded by P1. The process of making and reversing the equation by the subject is based on Reversible-Thinking Task intended for category 1 as follows.

- |   |  |
|---|--|
| <p>The process of making equation :</p> <ul style="list-style-type: none"> <li>- First, the subject focuses on the initial equation,</li> <li>- The subject then focuses on the identified element,</li> <li>- Next, the subject moves the identified element so that it results in new equations.</li> </ul> | <p>The process of reversing equation :</p> <ul style="list-style-type: none"> <li>- First, the subject focuses on the moved element,</li> <li>- Then, he reverses the operant side to the initial position in order to make the new equation equivalent with the initial one.</li> </ul> |
|---|--|

**Determining the value of unknown element**

The equation classified into this category is coded by P2. The process of making and reversing the equation by the subject is based on Reversible-Thinking Task intended for category 2 as follow.

- |  |   |
|--|---|
| <p>The process of making equation :</p> <ul style="list-style-type: none"> <li>- First, the subject pays intention to the initial equation,</li> <li>- Second, the focus is shift into the subject's first equation, which is a = 16 - 24</li> </ul> | <p>The process of reversing equation :</p> <ul style="list-style-type: none"> <li>- First, the subject focuses on -8,</li> <li>- Second, he changes -8 into 16 - 24 since -8 derives from 16 - 24,</li> </ul> |
|--|---|

- Third, determining the result of  $16 - 24$ , it considers  $a = -8$  as the subject's equation.
- Third, the subject moves the element 24 so that it comes into the initial equation.

**Operating the two sides of the initial equation with the same number**

The equations classified into this category are coded by P26, P27, and P30. The process of making and reversing the equations by the subject is based on Reversible-Thinking Task for intended for category 3 as follow.

The process of making equation :

- First, the subject focuses on the initial equation,
- Next, he operates the both sides of the initial equation with the same number.

The process of reversing equation :

- First, the subject takes note of the equation he made,
- Next, he re-operates the two sides of the equation with the same number. However, it is the inverse of the previous one used for making the equation.

**Operating the two sides the initial equation with equal expression**

The equation classified into this category is coded by P28. The process of making and reversing equation by the subject is based on Reversible-Thinking Task intended for category 4 as follow.

The process of making equation :

- First, the subject concerns on the initial equation,
- Next, the subject operates the two sides of the initial equation with an equal integer or expression (a combination consisting of operand number and arithmetical operation without considering the symbol "=") which has the same value.

The process of reversing equation :

- First, the subject takes note of the equation he has made,
- Second, he determines the result of operation derived from an expression; for instance, it results in  $f$
- Third, he operates the two sides of the equation using  $f$ , but it is the inverse of the previous one used for making the equation.

**Pointing to the equation on category 1, and then changing the identified constructing element**

The equations classified into this category are coded by P34 and P35. The process of making and reversing the equations by the subject is based on Reversible-Thinking Task intended for category 5 as follow.

The process of making equation :

- First, the subject points to the initial equation he previously made, which is  $a = 16 - 24$ , reasoning that it derives from the initial equation,
- Second, he concerns on the identified constructing elements on the first equation he made,
- Third, he changes one or both of the identified constructing elements into an expression which result is equal to the changed constructing element.

The process of reversing equation :

- First, the subject concerns on expression he made
- Second, he reconsiders the result of expression he made so that he may reverse the equation he successfully made into the initial condition. thus, it results in  $a = 16 - 24$
- Third, the subject moves the element 24 so that it reaches back to the initial condition.

**Pointing to the value of unknown element and then changing it into an expression**

The equations classified into this category are coded by P5, P6, P7, P8, P9, P29 and P31. The process of making and reversing the equations by the subject is based on Reversible-Thinking Task intended for category 6 as follow.

The process of making equation:

- First, the subject concerns on the equation he made, that is  $a = -8$ . SLT points to  $a = -8$  since it is equal to the initial equation and  $a = -8$  derives from the initial equation,

The process of reversing equation:

- First, the subject concerns on the expression he made,
- Second, he determines the result of the expression, so that it is found  $a = -8$ ,
- Third, he changes  $-8$  into  $16 - 24$  by reasoning that he attempts to correspond the identified constructing elements on the initial equation.

- Next, the subject changes  $-8$  into an **expression** which value is  $-8$ .
- Hence, since element 16 and 24 are identified, the subject finds himself easier to go back to the initial equation,
- Last, the subject moves the element 24 so it leads to the initial equation.

**Pointing to the initial equation, and then changing the identified constructing element**

The equations classified into this category are coded by P11, P12, P18, P20, P21, P22, P23, P24, P25, and P36. The process of making and reversing the equations by the subject is based on Reversible-Thinking Task intended for category 7 as follow.

- |   |  |
|---|--|
| <p>The process of making equation :</p> <ul style="list-style-type: none"> <li>- First, the subject concerns on the identified element on the initial equation,</li> <li>- Next, he changes one or both sides of the identified elements on the initial equation into an expression consisting of an operant and arithmetical operation, and when it is operated, the resulted value will equal to the changed identified element.</li> </ul> | <p>The process of reversing equation :</p> <ul style="list-style-type: none"> <li>- First, the subject concerns on the expression consisting of an operant and arithmetical operation he made.</li> <li>- Next, he reverses the expression into the original element by determining the result of the operation, with a reason that it allows him to turn back into the initial equation.</li> </ul> |
|---|--|

**Making any kinds of equations in case that the value of the unknown element equals to the value of the unknown element on the initial equation**

The equations classified into this category are coded by P10, P13, P14, P15, P16, P17, P19, P32, and P33. The process of making and reversing the equations by the subject is based on Reversible-Thinking Task intended for category 8 as follow.

- |   |   |
|---|---|
| <p>The process of making equation :</p> <ul style="list-style-type: none"> <li>- First, the subject constructs an element.</li> <li>- Next, he operates the element with another element and determines the result.</li> <li>- Then, he, again, operates it with another element and determines the result.</li> <li>- This operation will stop if the subject is willing to stop, in case that the value of a is <math>-8</math>.</li> </ul> | <p>The process of reversing equation</p> <ul style="list-style-type: none"> <li>- The subject changes the equation on category 8 into <math>a = -8</math>. His reason changing the equation into <math>a = -8</math> is to make him easier to reverse the equation into the initial one.</li> <li>- Next, the subject changes <math>-8</math> into <math>16 - 24</math>, it is because he corresponds to the elements on the initial equation which contains element 24 and 16; hence, it allows the subject to easily go back to the initial equation.</li> <li>- Next, the subject moves the element 24 so that it may lead to the initial equation.</li> </ul> |
|---|---|

**Using the definition of subtraction**

The equation classified into this category is coded by P3. The process of making and reversing the equation by the subject is based on Reversible-Thinking intended for category 9 as follow.

- |   |  |
|---|--|
| <p>The process of making equation :</p> <ul style="list-style-type: none"> <li>- First, the subject concerns on the first equation he made (<math>a = 16 - 24</math>).</li> <li>- Next, he changes <math>a = 16 - 24</math> into <math>a = 16 + (-24)</math> with a reason that a subtraction is in front of the element 24. Thus, the sign "-" belongs to 24.</li> <li>- Next, the subject changes <math>a = 16 + (-24)</math> into <math>a = -24 + 16</math> with a reason that <math>16 + (-24) = -24 + 16</math> (is commutative on addition).</li> <li>- Next, the subject writes down <math>a = -24 + 16</math> in the form of <math>-24 + 16 = a</math> with a reason that there is an sign (=), thus, he thinks that <math>a = -24 + 16</math> equals to <math>-24 + 16 = a</math></li> </ul> | <p>The process of reversing equation:</p> <ul style="list-style-type: none"> <li>- First, the subject concerns on <math>-24</math></li> <li>- Then, he moves the side of element <math>-24</math>, so that it turns into <math>16 = a + 24</math></li> <li>- Next, the subject changes <math>16 = a + 24</math> into <math>a + 24 = 16</math> with a reason that there is a sign (=), thus, the subject thinks that <math>16 = a + 24</math> equals to <math>a + 24 = 16</math>.</li> <li>- Next, the subject changes <math>a + 24</math> into <math>24 + a</math>, thus, the initial equation <math>24 + a = 16</math> is reached.</li> </ul> |
|---|--|

**Using commutative feature**

The equation classified into this category is coded by P4. The process of making and reversing the equation by the subject is based on Reversible-Thinking Task intended for category 10 as follow.

- |  |   |
|--|---|
| <p>The process of making equation :</p> <ul style="list-style-type: none"> <li>- First, the subject concerns on the left side of the initial equation,</li> <li>- Next, he changes <math>24 + a</math> into <math>a + 24</math> with a reason that the operation of addition is commutative, that is <math>24 + a = a + 24</math></li> </ul> | <p>The process of reversing equation :</p> <ul style="list-style-type: none"> <li>- Fisrt, the subject concerns on the left side of the initial equation,</li> <li>- Next, the subject reverses <math>a + 24</math> into <math>24 + a</math></li> </ul> |
|--|---|

Based on Table 1, what made the researcher felt interested the most was category 11. When making a new equation, the students used trial and error. However, the primary graders still involved logics in making equation, hence, all the equations they made were equivalent with the initial one. In relation to trial and error strategy, NCTM claimed, early efforts at justification by young children will involve trial-and-error strategies or the unsystematic trying of many cases (NCTM, 2000). Furthermore, one initial effort for students to solve problems is through trial and error strategy. In addition, NCTM describes that through trial and error, students have chances to explore their knowledge by trying all the cases. Hence, through such exploration, the students may acquire an understanding on appropriate and systematical procedures to solve problems.

Moreover, the result shows that there are 10 categories the students used to make and reverse their new equations into the initial one. The new equation is made by the subject. Among those 10 categories, there are 7 categories in which the subject goes through the same method in ways to make and reverse the new equation into the initial one. These seven categories include category 1, 2, 3, 5, 6, 7, and 10. Krutetskii argued, in reversing one’s thought process, it is not necessary to always travel through the same route but what matters is the order (i.e., in a reverse problem one starts from the output to the input (Krutetskii, 1976). It means that in order to revert to the initial data, it does not need to use the same method. Particularly, the activities of making and reversing the new equation to the initial condition through the same method are presented by the following Table 3.

**Table 3.** Activity of Making and Reversing the New Equation to the Initial Equation

Category	Aspects of Mathematical Reversible Thinking	
	Activities of making equation	Activities of reversing equation
1	Moving the side of the unknown element	Moving the operant side back to the initial position.
2	Determining the result of $16 - 24$ , hence it acquires $-8$	Changing $-8$ into $16 - 24$ .
3	Operating the two sides of the initial equation with the same number.	Operating the two sides of the equation back to the initial condition with the same number. However, the operation is the inverse of the operation used for making new equation.
5	Changing one or both of identified constructing elements on the initial equation into an expression which result equals to the changed identified constructing element.	Re-determining the result of operation derived from an expression the subject made.
6	Changing $-8$ into an expression which value is $-8$ .	Changing the expression made by the subject into $-8$ by determining the result of the expression so that it results in $a = -8$
7	Changing one or both of identified elements on the initial equation into an expression consisting of an operant and arithmetical operation, and when it is operated, the value will equals to the changed identified element.	Changing the expression the subject has successfully made into the initial condition by determining the result of operation from the expression.
10	Changing $24 + a$ into $a + 24$ with a reason that the operation of addition is commutative, that is $24 + a = a + 24$	Reversing $a + 24$ into $24 + a$



## Discussion

Activities of making equation and activities of reversing equation, if they were related to the research of Maf'ulah, Juniati, and Siswono in 2017, then activities of making equation was called a forward process while activities of reversing equation was called a reverse process (Maf'ulah et al., 2017). Actually, in reversible thinking, there were indeed two processes that are opposite each other, i.e. doing action and undoing action or working backwards (Maf'ulah et al., 2019; S. Maf'ulah & Juniati, 2019; Ajay Ramful, 2015). Working backwards is a particularly useful method in problem solving situations when the end result of a problem is known and one has to find the initial quantity. In step of problem solving by Polya, it was namely looking back (Polya, 1973). Reversible thinking is usually synonymous with problems related to inverses (Ikram et al., 2020; Maf'ulah & Juniati, 2020; Maf'ulah & Juniati, 2020, 2021). However, this research shows that to reveal reversible thinking, it did not had to go through problems related to inverses, the most important thing was that problems could involve reverse activity or working backwards. Category 1, which is constructed by the subject, refers to the category which characteristic is moving the constructing element. What should notice here is the term "moving the side". In teaching mathematics, such term is not applied and the subject should not move the constructing element. Rather, what it is supposed to be is adding the two sides of the initial equation ( $24 + a = 16$ ) with the inverse from the constructing element 24, hence, the left side will result in 0 [ $(-24) + 24 = 0$ ]. In category 1, there was equation (i.e. P1) that involved negation or invers, because the subject used inversion toward the related operation in his way making equations (Maf'ulah et al., 2017). Negation or invers and reciprocity were aspects of students' reversible thinking (Maf'ulah et al., 2017; Saparwadi et al., 2020). Both category 2 and category 8 imply an understanding about the meaning of variable. Variables are used in three ways in elementary school mathematics. They are used to represent unknowns, to represent quantities that vary, and to generalize properties. It indicates that on the primary graders, variables may play 3 roles including presenting the unknown element, presenting the different quantities, and generalizing a feature. In this case, the subject defines variable as the element which values is not identified yet (Greenes, 2004).

The subject claims that, on category 3 and 4, if the two sides of equation are operated with the same number or the same value, it will not impact on that equation. However, it is not allowed for the right and the left sides of the equation to be operated with different numbers since it will make the new equation unequal to the initial one. Such reason implies the subject's understanding on equality sign "=" indicating that the two sides of the equation are equal in their value. As Burns argued, one of initial algebraic concept the primary graders should receive is understanding the equal sign as an indication that quantities have the same value, not as a signal to write the answer (Fadillah & Jamilah, 2016; Ranti et al., 2017; Sari et al., 2016). Both category 3 and 4 show that in reversing a new equation to the initial equation, the subject uses a reciprocity or compensation. As Piaget suggested that one way to revert to the initial condition is through reciprocity (Inhelder & Piaget, 1958). Reciprocity links to a compensation or equivalent relationships. In this case, the male subject has high mathematical competence to operate the two sides of equation with the same element. As described by Adi that, in education field, he uses the concept of negation and compensation to solve the equation:  $14 - \frac{15}{7-x} = 9$ . In order to identify the value of  $x$ , the student may involve negation and reciprocity. Thus, he multiplies the two sides of the equation with  $7 - x$ , and it results in  $98 - 14x - 15 = 63 - 9x$  (Ramful, 2009). Category 9 shows that the primary students on the fifth grade have already understood the definition of subtraction. Following Billstein, Libeskind, & Lott, the operation of subtraction is defined that "for all integers  $a$  and  $b$ , it applies  $a - b = a + (-b)$ ". On the other hand, category 10 shows that the primary students on the fifth grade have already understood a commutative feature on the operation of addition (Billstein et al., 1990). The finding of this research shows that the mathematical reversible thinking of the primary students on the fifth grade and are in about 10 years old have evolved on abstract things. Particularly, the primary graders' thought points to the mathematical reversible thinking on formal stage since it does not only develop on concrete things but also abstract ones. This is inconsistent with a theory stating that the reversible thinking of the students with ages between 7 and 11 years old is on the concrete stage (Inhelder & Piaget, 1958).

## 4. CONCLUSION

Based on the findings and discussion of this study, it concludes that if the primary graders were provided with a simple equation and then asked to make and reverse the new equations equivalent with the initial one, it may reveal 10 categories of mathematical reversible thinking which include; (1)carrying the constructing element to the initial equation; (2)determining the unknown element; (3)operating the two sides of the initial equation with the same number; (4)operating the two sides of the initial equation with equal expression; (5)making equations based on the previous one and then changing the identified constructing element; (6)category which characteristic points to the value of the unknown element and

then changes the value into expression; (7)changing the building elements known in the original equation; (8) refers to the value of the initial equation variable; (9)using the definition of subtraction; and category; (10)using commutative feature on the operation of addition.

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